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EXISTENCE OF SOLUTION TO FRACTIONAL ORDER DELAY DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. In this manuscript, we consider the mathematical model of fractional functional differential equations of order $\alpha \in (2,3)$ with impulsive effects. We apply fixed point theorems to study the necessary and sufficient conditions of solution to this model. We provide a numerical example to illustrate the conditions and conclusion.

Keywords: *functional order differential equation with fractional derivative, initial conditions, differential equations with impulse, fixed point theorems.*

AMS Subject Classification: *34K37, 34A12, 34A37, 47H10.*

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1. Introduction

It is seen that the theory of fractional differential equations in infinite dimensional spaces have important role in several fields of sciences and technology like physics, chemistry, control, mechanics etc. One can see the cited [8, 11, 12, 13] monographs for more detail of this topic. Many dynamical systems such as remote control, implicit equations like Wheeler-Feynman equations, structured populations model which involve threshold phenomena etc. modelled in Fractional differential equations with infinite delay. From the application point of view these models are very useful. Due to this reason, we refer the papers [1, 4, 7, 9] and references therein for further study. Dynamical system with impulsive effects are paid attention by many authors because the model which are subjected to abrupt or instance changes are not described only by initial value problems but actually modeled in term of impulsive condition [5, 6, 10, 14]. These models are founded in the ecology, mechanics, electrical, medicine biology and others fields.

In the literature [5], it seen that Feckan *et al.* gave a very important concept about fractional differential equations with impulsive effects and present a counterexample to concur the correct formula. Based on corrects formula authors [5, 10] established the existence of solution for impulsive differential equations with fractional order $q \in (0,1)$ by using the fixed point theorem. On the other side some existence and uniqueness of solutions for impulsive fractional differential equation order $q \in (1,2)$ studied by [6, 14]. Recently, author [2] studied the nonlinear fractional differential equations of order $q \in (2,3)$ with integral boundary value conditions. We motivated by the papers [2, 6, 10, 11, 14], we consider following impulsive fractional functional integro differential equation of the form:

$$ {}^{C}D_t^{\alpha} y(t) = \mathrm{I}_t^{3-\alpha} f\big(t, y_t, B(y)\big),\ t \neq t_k \in [0,T], k = 1,2,..m \quad (1) $$

$$ y(t) = \phi(t),\ \ y'(0) = \varphi, y''(0) = \eta, t(-\infty, 0], \quad (2) $$

$$ \Delta y(t_k) = I_k\big(y(t_k^-)\big), \qquad \Delta y'(t_k) = J_k\big(y(t_k^-)\big), \\ \Delta y''(t_k) = L_k\big(y(t_k^-)\big), \quad (3) $$

where ${}^{C}D_t^{\alpha}$; $\mathrm{I}_t^{3-\alpha}$ denote Capoto's derivative of order α ∈ (2, 3); Riemann-Liouville integral of order 3-α > 0 and $f\big(t, y_t, B(y)\big)$ is a nonlinear function defined on ([0, T]$, \mathfrak{B}_h, \mathbb{X}$) to $\mathbb{X}$, where $\mathfrak{B}_h$ is a abstract phase space and $y_t(\theta) = y(\theta + t), \theta \in (-\infty, 0)$. The integro term $By(t) = \int_0^t K(t,s)y(s)\, ds$, where $K \in C(D, \mathbb{R}^+)$ with $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Functions y';y'' denote the ordinary derivatives of y with respect to t and $I_k, J_k, L_k, : \mathbb{X} \to \mathbb{X}$ are jump function with $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$; $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-)$; $\Delta y''(t_k) = y''(t_k^+) - y''(t_k^-)$ represents the right and left hand limits of $y(t)$ at $t = t_k$ such that $y(t_k^-) = y(t_k)$. For impulsive points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.

We have concerned with the existence result for the problem (1)-(3) which comes up from fluid mechanics and physics like Abraham-Lorentz force model of a non-relativistic, classical radiating, charged particle arises in loss of energy due to radiation. To the best of author's knowledge this work is new.

Rest of this work has four sections, second section provides some basic preliminaries facts. Third section contains the main result and fourth section has an example and conclusion.

2. Preliminaries

Consider a complex Banach space of function equipped with supremum norm $\|\,.\,\|_{\mathbb{X}}$ and abstract phase spaces $\mathfrak{B}_h$and $\mathfrak{B}'_h$, which are defined in [3] for functional differential equation with the infinite delay. Definitions used in this paper we refer books [8, 11, 12, 13] such as Riemann Liouville integral operator, Caputo's derivative and others preliminaries. For the analysis and to obtain the main results, we need the following consequences and hypothesis from the paper [3].

If $y : (-\infty, T] \to \mathbb{X}, T > 0$ is such that $y_0(t) = \phi(0) \in \mathbb{X}$, then for all $t \in (t_k, t_{k+1}]$, the following conditions hold:

1. $y_t \in \mathfrak{B}_h$
2. $\|y\| \leq H\|y\|_{\mathfrak{B}_h}$, where $H > 0$.
3. $\|y_t\|_{\mathfrak{B}_h} \leq C_1 sup_{0<s<t}\|y\| + C_2(t)\|\phi\|_{\mathfrak{B}'_h}$,

where $C_1; C_2 : [0,\infty) \to [0,\infty)$ are continuous functions and independent of $y(.)$. And also C_2 is locally bounded.

Definition 1. The Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f: [a, \infty) \to \mathbb{R}$ is defined as

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ t > a \geq 0,$$

where $n-1 < \alpha < n, n \in \mathbb{N}$ and $\Gamma(.)$ is Gamma function.

Definition 2. The Riemann- Liouville fractional (R-L) integral operator of order $\alpha > 0$ with lower limit a, for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$\mathrm{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \ t > a \geq 0.$$

Now, consider the linear fractional order Cauchy problem with impulsive effects to obtain the integral form of solution.

Lemma 1. If f *satisfies* the uniform Holder condition with the exponent $\beta \in (0, 1]$; then a solution $y(t), t \in (t_k, t_{k+1}]$ of a Cauchy problem

$${}^{C}D_t^{\alpha} y(t) = \mathrm{I}_t^{3-\alpha} f(t), \ t \in [0, T], t \neq t_k \ k = 1,2, \dots, m, \tag{4}$$

$$y(0) = y_0, \ y'(0) = y_1, y''(0) = y_2, \tag{5}$$

$$\Delta y(t_k) = I_k\big(y(t_k^-)\big), \Delta y'(t_k) = J_k\big(y(t_k^-)\big), \Delta y''(t_k) = L_k\big(y(t_k^-)\big), \tag{6}$$

is a solution if it satisfies following integral equation

$$\begin{aligned} y(t) = {} & y_0 + y_1 t + y_2 \frac{t^2}{2} + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big) + \sum_{0<t_k<t} (t-t_k) J_k\big(y(t_k^-)\big) \\ & + \sum_{0<t_k<t} \frac{(t-t_k)^2}{2} L_k\big(y(t_k^-)\big) + \sum_{0<t_{k-1}<t} \frac{1}{2} \int_{t_{k-1}}^{t_k} (t_k - s)^2 f(s) ds \\ & + \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s) f(s) ds + \sum_{0<t_k<t} \frac{(t-t_k)^2}{2} \int_{t_{k-1}}^{t_k} f(s)\, ds \\ & + \frac{1}{2} \int_{t_k}^{t} (t-s)^2 f(s) ds. \end{aligned} \tag{7}$$

Proof. If $t \in [0, t_1]$ then by applying the R-L integral on Eq. (4), we get

$$y(t) = a_0 + b_0 + \frac{c_0 t^2}{2} + \frac{1}{2} \int_0^t (t-s)^2 f(s) ds \tag{8}$$

where a_0, b_0 *and* c_0 are arbitrary constants. Using Eq. (5) we get,

$$y(t) = y_0 + y_1 t + y_2 \frac{t^2}{2} + \frac{1}{2} \int_0^t (t-s)^2 f(s) ds. \tag{9}$$

If $t \in (t_1, t_2]$, then by applying the R-L integral on Eq. (4), we have

$$y(t) = a_1 + b_1(t - t_1) + \frac{c_1 (t-t_1)^2}{2} + \frac{1}{2} \int_{t_1}^t (t-s)^2 f(s) ds \tag{10}$$

where a_1, b_1 *and* c_1 are again arbitrary constants. Using by impulsive condition $y'(t_k^+) - y'(t_k^-) = I_1$on Eq. (9) and (10), we get,

$$a_1 = y_0 + y_1 t_1 + y_2 \frac{{t_1}^2}{2} + \frac{1}{2} \int_0^{t_1} (t_1 - s)^2 f(s) ds. \tag{11}$$

From (10) and (11) we get

$$y(t) = y_0 + y_1 t_1 + y_2 \frac{{t_1}^2}{2} + I_1 + b_1 (t - t_1) + \frac{c_1 (t-t_1)^2}{2}$$

$$+\frac{1}{2}\int_0^{t_1}(t_1-s)^2 f(s)ds+\frac{1}{2}\int_{t_1}^{t}(t-s)^2 f(s)ds \tag{12}$$

On differentiating Eq. (9) and (10) with respect to t and applying impulsive condition $y'(t_k^+)-y'(t_k^-)=J_1$, we get

$$b_1 = y_1+y_2t_1+J_1+\int_0^{t_1}(t_1-s)\,f(s)ds. \tag{13}$$

From Eq. (12) and (13), we get

$$y(t) = y_0+y_1t+y_2\left(tt_1-\frac{{t_1}^2}{2}\right)+I_1+(t-t_1)J_1+\frac{c_1(t-t_1)^2}{2}$$
$$+(t-t_1)\int_0^{t_1}(t_1-s)\,f(s)ds+\frac{1}{2}\int_0^{t_1}(t_1-s)^2 f(s)ds$$
$$+\frac{1}{2}\int_{t_1}^{t}(t-s)^2 f(s)ds \tag{14}$$

Now, on double differentiating Eq. (9) and (10) with respect to t and impulsive condition $y''(t_k^+)-y''(t_k^-)=L_1$, we get

$$c_1 = L_1+y_2+\int_0^{t_1}f(s)ds \tag{15}$$

From Eq. (11) and (15), we get

$$y(t) = y_0+y_1t+y_2\frac{t^2}{2}+I_1+(t-t_1)J_1+\frac{(t-t_1)^2L_1}{2}$$
$$+\frac{1}{2}\int_0^{t_1}(t_1-s)^2 f(s)ds+(t-t_1)\int_0^{t_1}(t_1-s)\,f(s)ds+$$
$$\frac{(t-t_1)^2}{2}\int_0^{t_1}f(s)ds+\frac{1}{2}\int_{t_1}^{t}(t-s)^2 f(s)ds. \tag{16}$$

Repeating the process in this way and summarizing, we can get the result of equation (7) for $t\in(t_k,t_{k+1}]$. It is clear that the solution given in (7) satisfies the problem (4)-(6). This completes the proof of the lemma.

Now, we present the definition of solution to system (1)-(3) with the help of Lemma 1.

Definition 3. A function $y(t):(-\infty,T]\to\mathbb{X}$ such that $y(0)=\phi(0)$ is a solution of the problem (1)-(3) if and only if it satisfies the following integral equation

$$y(t)=\phi(0)+\varphi t+\eta\frac{t^2}{2}+\sum_{0<t_k<t}I_k\big(y(t_k^-)\big)+\sum_{0<t_k<t}(t-t_k)J_k\big(y(t_k^-)\big)$$
$$+\sum_{0<t_k<t}\frac{(t-t_k)^2}{2}L_k\big(y(t_k^-)\big)+\sum_{0<t_{k-1}<t}\frac{1}{2}\int_{t_{k-1}}^{t_k}(t_k-s)^2 f\big(s,y_s,B(y)\big)ds$$
$$+\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)\,f\big(s,y_s,B(y)\big)ds$$
$$+\sum_{0<t_k<t}\frac{(t-t_k)^2}{2}\int_{t_{k-1}}^{t_k}f\big(s,y_s,B(y)\big)\,ds+\frac{1}{2}\int_{t_k}^{t}(t-s)^2 f\big(s,y_s,B(y)\big)ds.$$

It is clear by a direct computation that the solution given in Definition 3 satisfies problem (1)-(3).

Theorem 1. (Banach fixed point theorem) Let $\mathcal{C}$ be a closed subset of a Banach space, and let $\mathcal{P}$: $\mathcal{C} \to \mathcal{C}$ be contraction mapping. Then $\mathcal{P}$ has a unique fixed point.

Theorem 2. (Schauder fixed point theorem) Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathbb{X}$, and let $\mathcal{P}$: $\mathcal{C} \to \mathcal{C}$ be continuous mapping with a compact image. Then $\mathcal{P}$ has a fixed point.

Theorem 3. (Ascoli-Arzela Theorem) Let $\mathcal{K}$ be a class of continuous functions defined over some interval J. Then $\mathcal{K}$ is relatively compact iff $\mathcal{K}$ is equi-continuous and uniformly bounded.

4. Existence results

In this section, we shall investigate the main result of problem (1)-(3). Let $f; I_k; J_k; L_k$ are jointly continuous functions and consider following extra assumptions on (1)-(3) to prove our results.

$\mathcal{A}_1$. There exist positive constants $\mathcal{L}_{f1}; \mathcal{L}_{f2}; \mathcal{L}_i; \mathcal{L}_j; \mathcal{L}_l$ such that

$$\|f(t,\phi,y) - f(t,\phi,z)\|_{\mathbb{X}} \le \mathcal{L}_{f1}\|\phi - \varphi\|_{\mathfrak{B}_h''} + \mathcal{L}_{f2}\|y - z\|_{\mathbb{X}};$$
$$\left\|I_k\big(y(t_k^-)\big) - I_k\big(z(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{L}_i\|y - z\|_{\mathbb{X}};$$
$$\left\|J_k\big(y(t_k^-)\big) - J_k\big(z(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{L}_j\|y - z\|_{\mathbb{X}};$$
$$\left\|L_k\big(y(t_k^-)\big) - L_k\big(z(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{L}_k\|y - z\|_{\mathbb{X}}, \phi, \varphi \in \mathfrak{B}_h; y, z \in \mathbb{X}.$$

$\mathcal{A}_2$. There exist positive constants $\mathcal{M}_f; \mathcal{M}_i; \mathcal{M}_j; \mathcal{M}_l$ such that

$$\|f(t,\phi,y)\|_{\mathbb{X}} \le \mathcal{M}_f;$$
$$\left\|I_k\big(y(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{M}_i;$$
$$\left\|J_k\big(y(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{M}_j;$$
$$\left\|L_k\big(y(t_k^-)\big)\right\|_{\mathbb{X}} \le \mathcal{M}_l, \phi \in \mathfrak{B}_h; y \in \mathbb{X}.$$

Now, we are in a position to state the existence theorem based on contraction principal Theorem 3.

Theorem 4. Let the assumption $\mathcal{A}_1$ hold and $\Pi < 1$, where

$$\Pi = m\mathcal{M}_i + mT\mathcal{M}_j + m\frac{T^2}{2}\mathcal{M}_l + (7m+1)\frac{T^3}{6}\left(\mathcal{L}_{f1}C_1^* + \mathcal{L}_{f2}B^*\right).$$

Then the problem (1)-(3) has a unique solution on [0, T].

Proof. Consider the space $\mathfrak{B}_h'' = \{y \in \mathfrak{B}_h' : \|y\| \le r\}$ which is a bounded, closed and convex subset and quipped with natural topology. Let us define an operator $\mathcal{O} : \mathfrak{B}_h'' \to \mathfrak{B}_h''$ as

$$\begin{aligned}\mathcal{O}y(t) = \phi(0) + \varphi t + \eta\frac{t^2}{2} + \sum_{0<t_k<t} I_k\big(\bar{y}(t_k^-)\big) + \sum_{0<t_k<t}(t - t_k)J_k\big(\bar{y}(t_k^-)\big)\\
+ \sum_{0<t_k<t}\frac{(t-t_k)^2}{2}L_k\big(\bar{y}(t_k^-)\big) + \sum_{0<t_{k-1}<t}\frac{1}{2}\int_{t_{k-1}}^{t_k}(t_k - s)^2 f\big(s, \bar{y}_s, B(\bar{y})\big)ds\\
+ \sum_{0<t_k<t}(t - t_k)\int_{t_{k-1}}^{t_k}(t_k - s)\, f\big(s, \bar{y}_s, B(\bar{y})\big)ds\end{aligned}$$

$$+\sum_{0<t_k<t}\frac{(t-t_k)^2}{2}\int_{t_{k-1}}^{t_k} f\big(s,\bar{y}_s,B(\bar{y})\big)\,ds$$
$$+\frac{1}{2}\int_{t_k}^{t}(t-s)^2 f\big(s,\bar{y}_s,B(\bar{y})\big)ds, \tag{17}$$

where $\bar{y}:(-\infty,T]\to\mathbb{X}$ is such that $\bar{y}=y$ on $[0,T]$. Now, we shall show that $\mathcal{O}$ has a unique fixed point in $\mathfrak{B}_h''$. For this purpose, we consider $y,y^*\in\mathfrak{B}_h''$, then

$$\|\mathcal{O}y-\mathcal{O}y^*\|_{\mathfrak{B}_h''}$$
$$\le\sum_{0<t_k<t}\big\|I_k\big(\bar{y}(t_k^-)\big)-I_k\big(\bar{y}^*(t_k^-)\big)\big\|_{\mathbb{X}}$$
$$+\sum_{0<t_k<t}|t-t_k|\big\|J_k\big(\bar{y}(t_k^-)\big)-J_k\big(\bar{y}^*(t_k^-)\big)\big\|_{\mathbb{X}}$$
$$+\sum_{0<t_k<t}\frac{|t-t_k|^2}{2}\big\|L_k\big(\bar{y}(t_k^-)\big)-L_k\big(\bar{y}^*(t_k^-)\big)\big\|_{\mathbb{X}}$$
$$+\sum_{0<t_{k-1}<t}\frac{1}{2}\int_{t_{k-1}}^{t_k}(t_k-s)^2\,\big\|f\big(s,\bar{y}_s,B(\bar{y})\big)-f\big(s,\bar{y}_s{}^*,B(\bar{y}^*)\big)\big\|_{\mathbb{X}}ds$$
$$+\sum_{0<t_k<t}|t-t_k|\int_{t_{k-1}}^{t_k}(t_k-s)\,\big\|f\big(s,\bar{y}_s,B(\bar{y})\big)$$
$$-f\big(s,\bar{y}_s{}^*,B(\bar{y}^*)\big)\big\|_{\mathbb{X}}ds$$
$$+\sum_{0<t_k<t}\frac{|t-t_k|^2}{2}\int_{t_{k-1}}^{t_k}\big\|f\big(s,\bar{y}_s,B(\bar{y})\big)-f\big(s,\bar{y}_s{}^*,B(\bar{y}^*)\big)\big\|_{\mathbb{X}}\,ds$$
$$+\frac{1}{2}\int_{t_k}^{t}(t-s)^2\,\big\|f\big(s,\bar{y}_s,B(\bar{y})\big)-f\big(s,\bar{y}_s{}^*,B(\bar{y}^*)\big)\big\|_{\mathbb{X}}ds.$$

Now, applying the assumptions $\boldsymbol{\mathcal{A}_1}$ and let $B^*=sup_{t\in[0,t]}\int_0^t K(t,s)\,ds<\infty$; $C_1^*=sup_{t\in[0,t]}C_1(t)$. Then we obtain

$$\|\mathcal{O}y-\mathcal{O}y^*\|_{\mathfrak{B}_h''}$$
$$\le m\mathcal{L}_i\|y-y^*\|_{\mathbb{X}}+mT\mathcal{L}_j\|y-y^*\|_{\mathbb{X}}+m\frac{T^2}{2}\mathcal{L}_l\|y-y^*\|_{\mathbb{X}}$$
$$+m\frac{T^3}{6}(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*)\|y-y^*\|_{\mathbb{X}}+m\frac{T^3}{2}(\mathcal{L}_{f1}C_1^*$$
$$+\mathcal{L}_{f2}B^*)\|y-y^*\|_{\mathbb{X}}+m\frac{T^3}{2}(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*)\|y-y^*\|_{\mathbb{X}}$$
$$+\frac{T^3}{6}(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*)\|y-y^*\|_{\mathbb{X}}$$
$$\le[m\mathcal{L}_i+mT\mathcal{L}_j+m\frac{T^2}{2}\mathcal{L}_l+m\frac{T^3}{6}\big(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*\big)$$
$$+m\frac{T^3}{2}\big(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*\big)+m\frac{T^3}{2}\big(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*\big)$$
$$+\frac{T^3}{6}\big(\mathcal{L}_{f1}C_1^*+\mathcal{L}_{f2}B^*\big]\|y-y^*\|_{\mathbb{X}}$$

$$\leq [m\mathcal{L}_i + mT\mathcal{L}_j + m\frac{T^2}{2}\mathcal{L}_l + (7m+1)\frac{T^3}{6}\left(\mathcal{L}_{f1}C_1^* + \mathcal{L}_{f2}B^*\right)] \, \|y - y^*\|_{\mathbb{X}}$$

$$\|\mathcal{O}y - \mathcal{O}y^*\|_{\mathfrak{B}_h''} \leq \Pi \|y - y^*\|_{\mathbb{X}}$$

Since Π <1, this implies that $\mathcal{O}$ is contraction map and has a unique fixed point by the Theorem 1, which is unique solution of problem (1)-(3). It is the proof of the theorem.

Theorem 5. Let the assumption $\boldsymbol{\mathcal{A}_2}$ hold then problem (1)-(3) has at least one solution.

Proof. We shall prove that operator $\mathcal{O}$ defined by (17) is continuous mapping with a compact image taking the help of following steps.

First, we show that is $\mathcal{O}$ continuous, for this propose, we consider a sequence $y^n \to y$ in $\mathfrak{B}_h''$, then

$$\begin{aligned}
&\|\mathcal{O}y^n - \mathcal{O}y\|_{\mathfrak{B}_h''} \\
&\leq \sum_{0<t_k<t} \left\|I_k\left(\bar{y}^n(t_k^-)\right) - I_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} \\
&+ \sum_{0<t_k<t} |t - t_k| \left\|J_k\left(\bar{y}^n(t_k^-)\right) - J_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} \\
&+ \sum_{0<t_k<t} \frac{|t - t_k|^2}{2} \left\|L_k\left(\bar{y}^n(t_k^-)\right) - L_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} \\
&+ \sum_{0<t_{k-1}<t} \frac{1}{2}\int_{t_{k-1}}^{t_k} (t_k - s)^2 \left\|f\left(s, \bar{y}_s{}^n, B(\bar{y})\right) - f\left(s, \bar{y}_s, B(\bar{y})\right)\right\|_{\mathbb{X}} ds \\
&+ \sum_{0<t_k<t} |t - t_k| \int_{t_{k-1}}^{t_k} (t_k - s) \left\|f\left(s, \bar{y}_s{}^n, B(\bar{y})\right)\right. \\
&\left. - f\left(s, \bar{y}_s, B(\bar{y})\right)\right\|_{\mathbb{X}} ds \\
&+ \sum_{0<t_k<t} \frac{|t - t_k|^2}{2} \int_{t_{k-1}}^{t_k} \left\|f\left(s, \bar{y}_s{}^n, B(\bar{y})\right) - f\left(s, \bar{y}_s, B(\bar{y})\right)\right\|_{\mathbb{X}} ds \\
&+ \frac{1}{2}\int_{t_k}^{t} (t - s)^2 \left\|f\left(s, \bar{y}_s{}^n, B(\bar{y})\right) - f\left(s, \bar{y}_s, B(\bar{y})\right)\right\|_{\mathbb{X}} ds.
\end{aligned}$$

Since the functions $f; I_k; J_k; L_k$ are continuous, then $\|\mathcal{O}y^n - \mathcal{O}y\|_{\mathfrak{B}_h''} \longrightarrow 0$ as n→∞. This indicate that $\mathcal{O}$ is continuous operator. Next, we shall prove that $\mathcal{O}$ maps bounded set into bounded set. For this, we take

$$\begin{aligned}
\|\mathcal{O}y(t)\|_{\mathfrak{B}_h''} &\leq \|\phi(0)\| + |\varphi||t| + |\eta|\frac{|t|^2}{2} + \sum_{0<t_k<t} \left\|I_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} \\
&+ \sum_{0<t_k<t} |t - t_k| \left\|J_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} + \sum_{0<t_k<t} \frac{|t - t_k|^2}{2} \left\|L_k\left(\bar{y}(t_k^-)\right)\right\|_{\mathbb{X}} \\
&+ \sum_{0<t_{k-1}<t} \frac{1}{2}\int_{t_{k-1}}^{t_k} (t_k - s)^2 \left\|f\left(s, \bar{y}_s, B(\bar{y})\right)\right\|_{\mathbb{X}} ds
\end{aligned}$$

$$+\sum_{0<t_k<t}|t-t_k|\int_{t_{k-1}}^{t_k}(t_k-s)\,\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}ds$$

$$+\sum_{0<t_k<t}\frac{|t-t_k|^2}{2}\int_{t_{k-1}}^{t_k}\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}\,ds+\frac{1}{2}\int_{t_k}^{t}(t-s)^2\,\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}ds$$

$$\leq \|\phi(0)\|+|\varphi|T+|\eta|\frac{T^2}{2}+m\mathcal{M}_i+mT\mathcal{M}_j+m\,\frac{T^2}{2}\mathcal{M}_l$$

$$+m\,\frac{T^3}{6}\mathcal{M}_f+m\,\frac{T^3}{2}\mathcal{M}_f+m\,\frac{T^3}{2}\mathcal{M}_f+\frac{T^2}{6}\mathcal{M}_f$$

$$\leq\ \|\phi(0)\|+|\varphi|T+|\eta|\frac{T^2}{2}+m\mathcal{M}_i+mT\mathcal{M}_j$$

$$+m\,\frac{T^2}{2}\mathcal{M}_l(7m+1)\mathcal{M}_f=\ C^*.$$

This is show that $\mathcal{O}$ maps bounded set into bounded set. Finally, we shall show that $\mathcal{O}$ is family of equicontinuous functions. To prove this step, let $e_1, e_2 \in [0,T]$ such that $0\leq e_1<e_2\leq T.$ Then, we

$$\|\mathcal{O}y(e_1)-\mathcal{O}y(e_2)\|_{\mathfrak{B}_h''}$$

$$\leq\ \varphi(e_1-e_2)+\eta\frac{(e_1-e_2)(e_1+e_2)}{2}$$

$$+\sum_{0<t_k<t}(e_1-e_2)\,\|J_k(\bar{y}(t_k^-))\|_{\mathbb{X}}$$

$$+\sum_{0<t_k<t}\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}\|J_k(\bar{y}(t_k^-))\|_{\mathbb{X}}$$

$$+\sum_{0<t_k<t}(e_1-e_2)\int_{t_{k-1}}^{t_k}(t_k-s)\,\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}ds$$

$$+\sum_{0<t_k<t}\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}\int_{t_{k-1}}^{t_k}\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}\,ds$$

$$+\left\|\frac{1}{2}\int_{t_k}^{e_1}(e_1-s)^2\,f(s,\bar{y}_s,B(\bar{y}))ds\right.$$

$$-\frac{1}{2}\int_{t_k}^{e_1}(e_2-s)^2\,f(s,\bar{y}_s,B(\bar{y}))ds$$

$$\left.-\frac{1}{2}\int_{e_1}^{e_2}(e_2-s)^2\,f(s,\bar{y}_s,B(\bar{y}))ds\right\|_{\mathbb{X}}$$

$$\leq\varphi(e_1-e_2)+\eta\frac{(e_1-e_2)(e_1+e_2)}{2}$$

$$+\sum_{0<t_k<t}(e_1-e_2)\,\|J_k(\bar{y}(t_k^-))\|_{\mathbb{X}}$$

$$+\sum_{0<t_k<t}\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}\|J_k(\bar{y}(t_k^-))\|_{\mathbb{X}}$$

$$
\begin{aligned}
&+\sum_{0<t_k\ <t}(e_1-e_2)\int_{t_{k-1}}^{t_k}(t_k-s)\,\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}ds\\
&+\sum_{0<t_k<t}\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}\int_{t_{k-1}}^{t_k}\|f(s,\bar{y}_s,B(\bar{y}))\|_{\mathbb{X}}\,ds\\
&+\frac{(e_1-e_2)}{2}\int_{t_k}^{e_1}(e_1+e_2-s)^2\,\|f(s,\bar{y}_s,B(\bar{y}))ds\|_{\mathbb{X}}\\
&+\frac{1}{2}\int_{e_1}^{e_2}(e_2-s)^2\,\|f(s,\bar{y}_s,B(\bar{y}))ds\|_{\mathbb{X}}\\
&\leq\varphi(e_1-e_2)+\eta\frac{(e_1-e_2)(e_1+e_2)}{2}+(e_1-e_2)m\mathcal{M}_j\\
&+\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}m\mathcal{M}_l\\
&+\sum_{0<t_k<t}(e_1-e_2)\mathcal{M}_f\int_{t_{k-1}}^{t_k}(t_k-s)\,ds\\
&+\sum_{0<t_k<t}\frac{(e_1-e_2)(e_1+e_2-2t_k)}{2}\mathcal{M}_f\int_{t_{k-1}}^{t_k}ds\\
&+\frac{(e_1-e_2)}{2}\mathcal{M}_f\int_{t_k}^{e_1}(e_1+e_2-2s)^2ds+\frac{1}{2}\mathcal{M}_f\int_{e_1}^{e_2}(e_2-s)^2\\
&\leq
\end{aligned}
$$

$$
\begin{aligned}
(e_1-e_2)\Bigg[&\varphi+\eta\frac{(e_1-e_2)(e_1+e_2)}{2}+(e_1-e_2)m\mathcal{M}_j+\frac{(e_1+e_2-2t_k)}{2}m\mathcal{M}_l\\
&+\sum_{0<t_k<t}\mathcal{M}_f\int_{t_{k-1}}^{t_k}(t_k-s)\,ds\\
&+\sum_{0<t_k<t}\frac{(e_1+e_2-2t_k)}{2}\mathcal{M}_f\int_{t_{k-1}}^{t_k}ds\\
&+\frac{1}{2}\mathcal{M}_f\int_{t_k}^{e_1}(e_1+e_2-2s)^2ds+\frac{1}{2}\mathcal{M}_f(e_1-e_2)^2\Bigg].
\end{aligned}
$$

It is clear that $\|\mathcal{O}y(e_1)-\mathcal{O}y(e_2)\|_{\mathfrak{B}_h^{''}}\longrightarrow 0$ as $e_1\to e_2$. This step proves that $\mathcal{O}y$ is a family of equi-continuous functions. Combining last two steps and by Theorem 3, $\mathcal{O}$ has compact image. Finally, it follows Theorem 2 that the operator $\mathcal{O}$ has affixed point which is the solution of the system (1)-(3).
It is the proof of the theorem

5. Example

Consider the following non-linear fractional functional differential equations with impulsive effects:

$$ {}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(3-\alpha)} \int_0^t \frac{(t-s)^{2-\alpha} y(t-\sigma(t))}{(49+y(t-\sigma(t)))} + $$
$$ + \int_0^t \cos(t-s) \frac{y}{49+y} ds \qquad (18) $$
$$ y(t) = \phi(t), y'(0) = \varphi, y''(0) = \eta, \quad t \in (-\infty, 0] \qquad (19) $$
$$ \Delta y|_{t=\frac{1}{2}} = \frac{y_{\frac{1}{2}}}{9 + y_{\frac{1}{2}}}, \Delta y'|_{t=\frac{1}{2}} = \frac{y_{\frac{1}{2}}}{16 + y_{\frac{1}{2}}}, \Delta y''|_{t=\frac{1}{2}} = \frac{y_{\frac{1}{2}}}{25 + y_{\frac{1}{2}}}. \qquad (20) $$

For the abstract phase space $\mathfrak{B}_h'$, let $\rho(t) = t - \sigma(0)$, and we have the following setting

$$ f(t.\phi, B(y)) = \frac{\phi}{49+\phi} + \int_0^t \cos\,(t-s) \frac{y}{49+y} ds $$

With the above setting, it can be seen that the assumptions of Theorem 4 are satisfied with the following calculation

$\|f(t.\phi, B(y)) - f(t.\varphi, B(z))\| \leq \frac{1}{49}\|\phi - \varphi\| + \frac{1}{49}\|y - z\|$;

$\|I_k(y(t_k^-)) - I_k(z(t_k^-))\| \leq \frac{1}{9}\|y - z\|$;

$\|J_k(y(t_k^-)) - J_k(z(t_k^-))\| \leq \frac{1}{16}\|y - z\|$;

$\|L_k(y(t_k^-)) - L_k(z(t_k^-))\| \leq \frac{1}{25}\|y - z\|$

Similarly, it can be seen for Theorem 5 as

$\|f(t.\phi, B(y))\| \leq \frac{2}{49}$; $\|I_k(y(t_k^-))\| \leq \frac{1}{9}$; $\|J_k(y(t_k^-))\| \leq \frac{1}{16}$; $\|L_k(y(t_k^-))\| \leq \frac{1}{25}$

We can see by above computations that system (18)-(20) have unique or at-least one solution.

6. Conclusion

In this article, we have considered fractional model of Abraham-Lorentz force (1)-(3) arise in physics. We have obtained the necessary and sufficient condition for existence of solution of model (1)-(3) with impulsive effects and with the help of Banach and Schauder fixed point theorems. Further, a numerical example offered to verify theorems**.**

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